The exterior Bitsadze-Lavrentjev problem for quaterelliptic-quaterhyperbolic equations in a doubly connected domain

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Abstract

The famous Tricomi equation was established in 1923, by F. G. Tricomi, who is the pioneer of parabolic elliptic and hyperbolic boundary value problems and related problems of variable type. In 1945, F. I. Frankl established a generalization of these problems for the well-known Chaplygin equation. In 1953 and 1955, M. H. Protter generalized these problems even further. In 1977, we generalized these results in several n-dimensional simply connected domains. In 1950-1951, M. A. Lavrentjev and A. V. Bitsadze investigated the Bitsadze - Lavrentjev equation. In 1990, we proposed the exterior Tricomi problem. In 2002, we considered uniqueness of quasi-regular solutions for a bi-parabolic elliptic bi-hyperbolic Tricomi problem. In 2006, G. C. Wen investigated the exterior Tricomi problem for general mixed type equations. In 2011, we established the exterior Tricomi and Frankl problems for quaterelliptic - quaterhyperbolic equations. In 2014, D. Amanov and J. M. Rassias investigated boundary value problems for the higher order generalized mixed-parabolic equation. In this paper we investigate the exterior Bitsadze-Lavrentjev problem for quaterelliptic -quaterhyperbolic Bitsadze-Lavrentjev PDEquations with eight parabolic lines in a doubly connected domain and propose open problems. These problems are of vital importance in fluid mechanics.

1 Introduction

In 1904, S. A. Chaplygin [11] pointed out that the nonlinear equation of an adiabatic potential perfect gas :

$$(\rho^2\alpha^2 - \psi_y^2)\psi_{xx} + 2\psi_x\psi_y\psi_{xy} + (\rho^2\alpha^2 - \psi_x^2)\psi_{yy} = 0,$$

is closely connected with the study of the linear mixed type equation

$$K(y)u_{xx} + u_{yy} = 0,$$

named Chaplygin equation , where $\psi = \psi(x, y)$ is the stream function, α =local velocity of sound and ρ =density of gas.

In 1923, F. G. Tricomi [19] initiated the work on boundary value problems for linear partial differential mixed type equations of second order and related equations of variable type. The well-known mixed type partial differential equation is called Tricomi equation :

$$yu_{xx} + u_{yy} = 0,$$

after F. G. Tricomi ,who introduced this equation, for functions $u = u(x, y)$ in a real (x, y)-region. It plays a central role in the mathematical analysis of the transonic flows, as it is of elliptic and hyperbolic type, where the coefficient y of the second order partial derivative of the involved function $u = u(x, y)$ with respect to x, changes sign. Besides this equation is of parabolic type, where y vanishes. In 1945, F. I. Frankl [3] drew attention to the fact that the Tricomi problem was closely connected to the study of gas flow with nearly sonic speeds. In 1953 and 1955, M. H. Protter [7] generalized and improved the afore-mentioned results in the euclidean plane. In 1977, we [8] generalized these results in R^n, $(n > 2)$. In 1950-1951, M. A. Lavrentjev and A. V. Bitsadze ([29]-[30]) investigated the famous Bitsadze-Lavrentjev equation:

$$sgn(y)u_{xx} + u_{yy} = 0,$$

where

$$sgn(y) = 1, y > 0; = 0, y = 0; = -1, y < 0.$$

In 1982, we [9] established a maximum principle of the Cauchy problem for hyperbolic equations in $R^{n+1}, (n \geq 2)$. In 1983, we [10] solved the Tricomi problem with two parabolic lines of degeneracy and, in 1992, we [12] established the well-posedness of the Tricomi problem in euclidean regions. Interesting results for the Tricomi problem were achieved by G. Baranchev [1] in 1986, and M. Kracht and E. Kreyszig [4] in 1986, as well. Related information was reported by G. Fichera [2] in 1985, and E. Kreyszig ([5]-[6]) in 1989 and 1994. Our ([11], [14]-[15]) work, in 1990 and 1999, was in similar areas of mixed type equations. In 1990-2009, G. C. Wen et al. ([17], [20]-[28], [31]) have applied the complex analytic method and achieved fundamental uniqueness and existence results for solutions of the Tricomi and Frankl problems for the classical mixed type partial differential equations with boundary conditions. In 1993, R.I. Semerdjieva [18] introduced the hyperbolic equation

$$K_1(y)u_{xx} + (K_2(y)u_y)_y + ru = f,$$

in the lower half-plane. In 1997, we [13] considered the more general case of the above hyperbolic equation, so that it was elliptic in the upper half-plane and parabolic on the line $y = 0$. In 2002, we [16] considered the more general Tricomi problem with partial differential equation the new bi-parabolic elliptic bi-hyperbolic equation

$$Lu = K_1(y) (M_2(x)u_x)_x + M_1(x) (K_2(y)u_y)_y + r(x, y)u = f(x, y), \qquad (1.1)$$

which is parabolic on both segments $x = 0, 0 < y \leq 1; y = 0, 0 < x \leq 1$, elliptic in the euclidean region $G_e = \{(x, y) \in G(\subset R^2) : x > 0, y > 0\}$ and hyperbolic in both regions

$$G_{h_1} = \{(x, y) \in G(\subset R^2) : x > 0, y < 0\}; G_{h_2} = \{(x, y) \in G(\subset R^2) : x < 0, y > 0\},$$

with G the mixed domain of (1.1). In 1999, we [15] proved existence of weak solutions for a particular Tricomi problem. Then we established uniqueness of quasi-regular solutions ([8], [10]-[13], [16]) for the Tricomi problem. In 2011, we ([32]-[33]) established the uniqueness of quasi-regular solutions for the exterior Tricomi and Frankl problems. In this paper we investigate the exterior Bitsadze-Lavrentjev problem for general Bitsadze-Lavrentjev quaterelliptic and quaterhyperbolic equations:

$$Lu = sgn (y(y - 1) |k(y)|) u_{xx} + sgn (x(x + 1) |m(x)|) u_{yy} + r(x, y)u = f(x, y), \qquad (1.2)$$

$$k = k(y) \neq 0; m = m(x) \neq 0,$$

everywhere in D, with eight parabolic lines of degeneracy in a doubly connected mixed domain and establish uniqueness of quasi-regular solutions. Also we propose new open problems. These results are interesting in Aerodynamics and Hydrodynamics. The Mixed type partial differential equations are encountered in the theory of transonic flow and they give rise to special boundary value problems, called the Tricomi and Frankl as well as Bitsadze-Lavrentjev problems. The Transonic flows involve a transition from the subsonic to the supersonic region through the sonic.

Definition 1.1. The Bitsadze-Lavrentjev problem or Problem BL consists of finding a function u which satisfies the afore-mentioned Bitsadze-Lavrentjev equation (1.2) in a mixed domain D :
a simply connected and bounded (x, y)-region by a rectifiable Jordan (non-self-intersecting) elliptic arc σ (for $y > 0$) with endpoints $O = (0, 0)$ and $A = (1, 0)$ and by two hyperbolic characteristics Γ, γ of the Bitsadze-Lavrentjev equation satisfying the pertinent characteristic equation such that these characteristics Γ, γ meet at a point P (for $y < 0$) with Γ emanating from $A = (1, 0)$ and γ from $O = (0, 0)$,

$$\Gamma : y = x - 1 \text{ and } \gamma : y = -x,$$

and u assumes prescribed continuous boundary values on both arcs σ and γ. The portion of D lying in the upper half-plane, above the $x - axis$, is the elliptic region; portion of D lying in the lower half-plane, below the $x - axis$, is the hyperbolic region; and the segment OA is parabolic.

Definition 1.2. A function $u = u(x, y)$ is regular solution of Problem BL if:

1. u is continuous in the closure of D which is the union of D with its boundary consisting of the three curves σ, Γ, γ;

2. The first order partial derivatives of u are continuous in the closure of D except points O, A, where they may have poles of order less than $2/3$;

3. The second order partial derivatives of u are continuous in D except possibly on OA where they may not exist;

4. u satisfies Bitsadze-Lavrentjev equation at all points of D except OA;

5. u assumes prescribed continuous boundary values on both arcs σ, γ;

2 Exterior Bitsadze-Lavrentjev problem: Tricomi Case

Consider the general Bitsadze-Lavrentjev quaterelliptic - quaterhyperbolic equation (1.2) with eight parabolic lines of degeneracy in a bounded doubly connected mixed domain D with a piecewise smooth boundary ∂D, where $f = f(x, y)$ is continuous in D, $r = r(x, y)$ is once-continuously differentiable in D, $K = K(y) := sgn(y(y - 1)|k(y)|)$ is discontinuous for $y \in [-k_1, k_2]$ with $-k_1 = inf\{y : (x, y) \in D\}$ and $k_2 = sup\{y : (x, y) \in D\}$, and $M = M(x) := sgn(x(x + 1)|m(x)|)$ is discontinuous for $x \in [-m_1, m_2]$ with $-m_1 = inf\{x : (x, y) \in D\}$ and $m_2 = sup\{x : (x, y) \in D\}$, so that $k = k(y) \neq 0$; $m = m(x) \neq 0$ everywhere in D. Besides

$$K = \begin{cases} +1 & for \quad \{y < 0\} \cup \{y > 1\} \\ 0 & for \quad \{y = 0\} \cup \{y = 1\} \\ -1 & for \quad \{0 < y < 1\} \end{cases} ; \quad M = \begin{cases} +1 & for \quad \{x < -1\} \cup \{x > 0\} \\ 0 & for \quad \{x = 0\} \cup \{x = -1\} \\ -1 & for \quad \{-1 < x < 0\} \end{cases} .$$
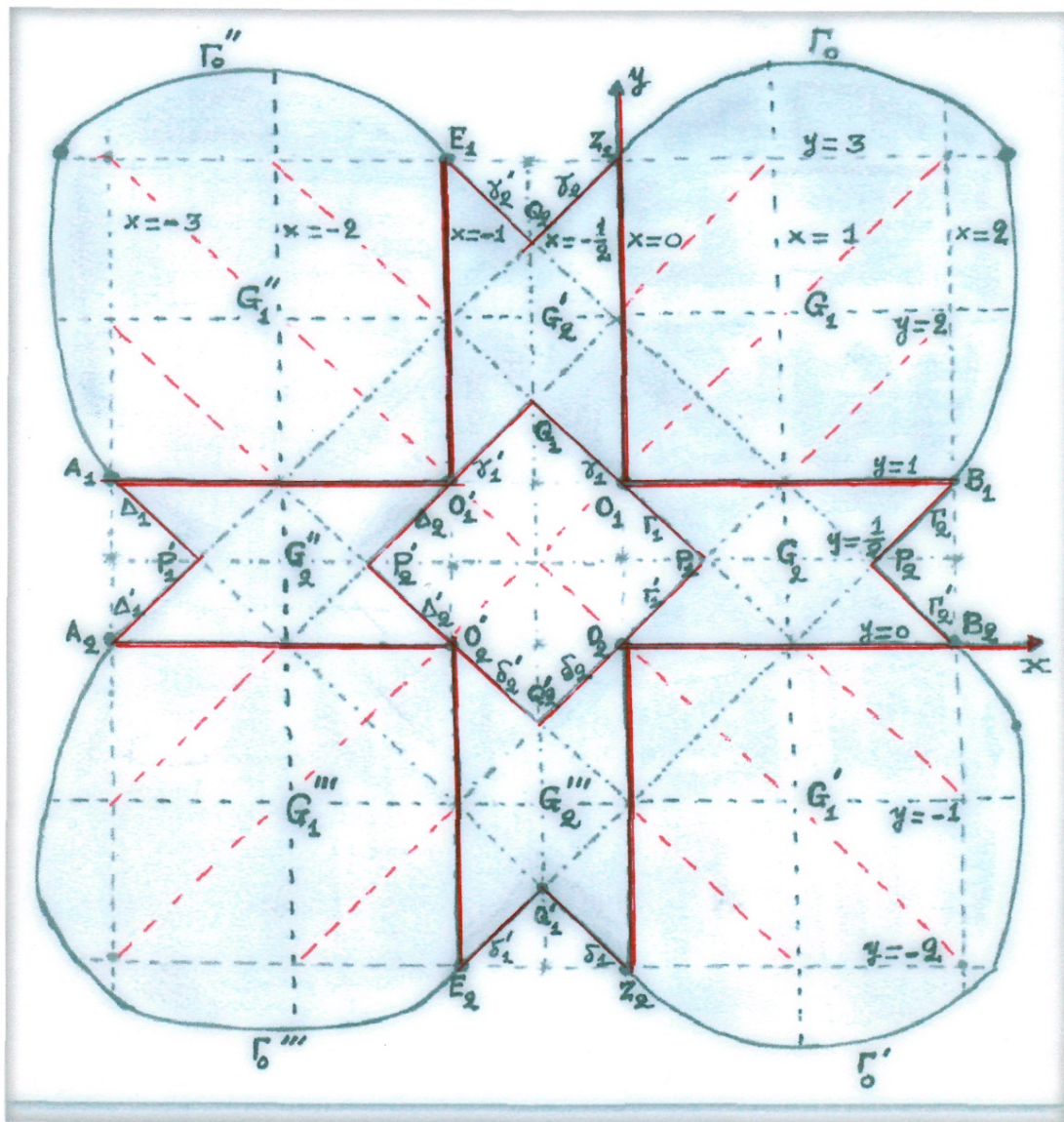
FIGURE 1. (Figure for Bitsadze-Lavrentjev problem: Tricomi case)

The boundary $\partial D = Ext(D) \cup Int(D)$ of the doubly connected domain D is formed by the following two exterior and interior boundaries

$$Ext(D) = (\Gamma_0 \cup \Gamma_0' \cup \Gamma_0'' \cup \Gamma_0''') \cup (\Gamma_2 \cup \Gamma_2') \cup (\gamma_2 \cup \gamma_2') \cup (\Delta_1 \cup \Delta_1') \cup (\delta_1 \cup \delta_1'),$$

$$Int(D) = (\Gamma_1 \cup \Gamma_1') \cup (\gamma_1 \cup \gamma_1') \cup (\Delta_2 \cup \Delta_2') \cup (\delta_2 \cup \delta_2') :$$

the interior boundary of D, which is the boundary of the square:

$$P_1 Q_1 P_2' Q_2' = P_1 O_1 Q_1 O_1' P_2' O_2' Q_2' O_2 P_1,$$

respectively: In the right hyperbolic domain $G_2 = \{(x,y) \in D : 0 < x < 2, 0 < y < 1\}$ with boundary $\partial G_2 = (O_1 B_1) \cup (O_2 B_2) \cup (\Gamma_1 \cup \Gamma_1') \cup (\Gamma_2 \cup \Gamma_2')$, where $O_1 B_1, O_2 B_2$ are two parabolic lines with end points $O_1 = (0,1), B_1 = (2,1)$ and $O_2 = (0,0), B_2 = (2,0)$ and $\Gamma_1, \Gamma_1', \Gamma_2, \Gamma_2'$ are four characteristics with end points: $O_1, B_1, O_2, B_2; P_1 = \left(\frac{1}{2}, \frac{1}{2}\right), P_2 = \left(\frac{3}{2}, \frac{1}{2}\right)$, so that:

$$\Gamma_1 = (O_1 P_1) : \int_0^x \sqrt{M(t)} dt = -\int_1^y \sqrt{-K(t)} dt \quad or \quad y = -x + 1,$$

$$\Gamma_1' = (P_1 O_2) : \int_0^x \sqrt{M(t)} dt = \int_0^y \sqrt{-K(t)} dt \quad or \quad y = x,$$

$$\Gamma_2 = (P_2 B_1) : \int_2^x \sqrt{M(t)} dt = \int_1^y \sqrt{-K(t)} dt \quad or \quad y = x - 1,$$

$$\Gamma_2' = (B_2 P_2) : \int_2^x \sqrt{M(t)} dt = -\int_0^y \sqrt{-K(t)} dt \quad or \quad y = -x + 2,$$

where $K = -1; M = 1$, such that on the characteristics: $H = -(dy)^2 + (dx)^2 = 0$. In the upper hyperbolic domain $G_2' = \{(x,y) \in D : -1 < x < 0, 1 < y < 3\}$ with boundary $\partial G_2' = (O_1 Z_1) \cup (O_1' E_1) \cup (\gamma_1 \cup \gamma_1') \cup (\gamma_2 \cup \gamma_2')$, where $O_1 Z_1, O_1' E_1$ are two parabolic lines with end points $O_1 = (0,1), Z_1 = (0,3)$ and $O_1' = (-1,1), E_1 = (-1,3)$ and $\gamma_1, \gamma_1', \gamma_2, \gamma_2'$ are four characteristics with end points: $O_1, Z_1, O_1', E_1; Q_1 = \left(\frac{-1}{2}, \frac{3}{2}\right), Q_2 = \left(\frac{-1}{2}, \frac{5}{2}\right)$, so that:

$$\gamma_1 = (Q_1 O_1) : \int_0^x \sqrt{-M(t)} dt = -\int_1^y \sqrt{K(t)} dt \quad or \quad y = -x + 1, \quad \gamma_1 \text{ is the extension of } \Gamma_1,$$

$$\gamma_1' = (O_1' Q_1) : \int_{-1}^x \sqrt{-M(t)} dt = \int_1^y \sqrt{K(t)} dt : \quad or \quad y = x + 2,$$

$$\gamma_2 = (Z_1 Q_2) : \int_0^x \sqrt{-M(t)} dt = \int_3^y \sqrt{K(t)} dt : \quad or \quad y = x + 3,$$

$$\gamma_2' = (Q_2 E_1) : \int\limits_{-1}^{x} \sqrt{-M(t)}dt = -\int\limits_{3}^{y} \sqrt{K(t)}dt \ \ or \ \ y = -x + 2,$$

where $K = 1; M = -1$, such that on the characteristics: $-H = (dy)^2 - (dx)^2 = 0$. In the left hyperbolic domain $G_2'' = \{(x, y) \in D : -3 < x < -1, 0 < y < 1\}$ with boundary $\partial G_2'' = (O_1' A_1) \cup (O_2' A_2) \cup (\Delta_1 \cup \Delta_1') \cup (\Delta_2 \cup \Delta_2')$, where $O_1' A_1, O_2' A_2$ are two parabolic lines with end points $O_1' = (-1, 1)$, $A_1 = (-3, 1)$ and $O_2' = (-1, 0)$, $A_2 = (-3, 0)$ and $\Delta_1, \Delta_1', \Delta_2, \Delta_2'$ are four characteristics with end points: $O_1', A_1, O_2', A_2; P_1' = \left(\frac{-5}{2}, \frac{1}{2}\right), P_2' = \left(\frac{-3}{2}, \frac{1}{2}\right)$, so that:

$$\Delta_1 = (A_1 P_1') : \int\limits_{-3}^{x} \sqrt{M(t)}dt = -\int\limits_{1}^{y} \sqrt{-K(t)}dt \ \ or \ \ y = -x - 2,$$

$$\Delta_1' = (P_1' A_2) : \int\limits_{-3}^{x} \sqrt{M(t)}dt = \int\limits_{0}^{y} \sqrt{-K(t)}dt \ \ or \ \ y = x + 3, \ \Delta_1' \ and \ \gamma_2 \ lie \ on \ A_2 Z_1,$$

$$\Delta_2 = (P_2' O_1') : \int\limits_{-1}^{x} \sqrt{M(t)}dt = \int\limits_{1}^{y} \sqrt{-K(t)}dt \ \ or \ \ y = x + 2, \ \ \Delta_2 \ is \ an \ extension \ of \ \gamma_1',$$

$$\Delta_2' = (O_2' P_2') : \int\limits_{-1}^{x} \sqrt{M(t)}dt = -\int\limits_{0}^{y} \sqrt{-K(t)}dt \ \ or \ \ y = -x - 1,$$

where $K = -1; M = 1$, such that on the characteristics: $H = -(dy)^2 + (dx)^2 = 0$. In the lower hyperbolic domain $G_2''' = \{(x, y) \in D : -1 < x < 0, -2 < y < 0\}$ with boundary $\partial G_2''' = (O_2 Z_2) \cup (O_2' E_2) \cup (\delta_1 \cup \delta_1') \cup (\delta_2 \cup \delta_2')$, where $O_2 Z_2, O_2' E_2$ are two parabolic lines with end points $O_2 = (0, 0), Z_2 = (0, -2)$ and $O_2' = (-1, 0), E_2 = (-1, -2)$ and $\delta_1, \delta_1', \delta_2, \delta_2'$ are four characteristics with end points: $O_2, O_2', E_2, Z_2; Q_1' = \left(\frac{-1}{2}, \frac{-3}{2}\right), Q_2' = \left(\frac{-1}{2}, \frac{-1}{2}\right)$, so that:

$$\delta_1 = (Q_1' Z_2) : \int\limits_{0}^{x} \sqrt{-M(t)}dt = -\int\limits_{-2}^{y} \sqrt{K(t)}dt \ \ or \ \ y = -x - 2, \ \delta_1 \ and \ \Delta_1 \ lie \ on \ A_1 Z_2,$$

$$\delta_1' = (E_2 Q_1') : \int\limits_{-1}^{x} \sqrt{-M(t)}dt = \int\limits_{-2}^{y} \sqrt{K(t)}dt \ \ or \ \ y = x - 1, \ \delta_1' \ and \ \Gamma_2 \ lie \ on \ B_1 E_2,$$

$$\delta_2 = (O_2 Q_2') : \int\limits_{0}^{x} \sqrt{-M(t)}dt = \int\limits_{0}^{y} \sqrt{K(t)}dt \ \ or \ \ y = x, \ \ \delta_2 \ is \ an \ extension \ of \ \Gamma_1',$$

$$\delta_2' = (Q_2' O_2') : \int\limits_{-1}^{x} \sqrt{-M(t)}dt = -\int\limits_{0}^{y} \sqrt{K(t)}dt \ \ or \ \ y = -x - 1, \ \ \delta_2' \ is \ an \ extension \ of \ \Gamma_1',$$

where $K = 1; M = -1$, such that on the characteristics: $-H = (dy)^2 - (dx)^2 = 0$. Note that $K = M = -1$, in the interior of the square $P_1Q_1P_2'Q_2'$, which contains no part of the mixed domain D.

In the upper right elliptic domain $G_1 = \{(x,y) \in D : x > 0, y > 1\}$ with boundary $\partial G_1 = (O_1B_1) \cup (O_1Z_1) \cup \Gamma_0$, where O_1B_1, O_1Z_1 are two parabolic lines with end points $O_1 = (0,1), B_1 = (2,1)$ and $O_1 = (0,1), Z_1 = (0,3)$ and Γ_0 is the upper right elliptic arc connecting points $B_1 = (2,1), Z_1 = (0,3)$.

In the lower right elliptic domain $G_1' = \{(x,y) \in D : x > 0, y < 0\}$ with boundary $\partial G_1' = (O_2B_2) \cup (O_2Z_2) \cup \Gamma_0'$, where O_2B_2, O_2Z_2 are two parabolic lines with end points $O_2 = (0,0), B_2 = (2,0)$ and $O_2 = (0,0), Z_2 = (0,-2)$ and Γ_0' is the lower right elliptic arc connecting points $B_2 = (2,0), Z_2 = (0,-2)$.

In the upper left elliptic domain $G_1'' = \{(x,y) \in D : x < -1, y > 1\}$ with boundary $\partial G_1'' = (O_1'E_1) \cup (O_1'A_1) \cup \Gamma_0''$, where $O_1'E_1, O_1'A_1$ are two parabolic lines with end points $O_1' = (-1,1), E_1 = (-1,3)$ and $O_1' = (-1,1), A_1 = (-3,1)$ and Γ_0'' is the upper left elliptic arc connecting points $A_1 = (-3,1)$ and $E_1 = (-1,3)$.

In the lower left elliptic domain $G_1''' = \{(x,y) \in D : x < -1, y < 0\}$ with boundary $\partial G_1''' = (O_2'A_2) \cup (O_2'E_2) \cup \Gamma_0'''$, where $O_2'E_2, O_2'A_2$ are two parabolic lines with end points $O_2' = (-1,0), E_2 = (-1,-2)$ and $O_2' = (-1,0), A_2 = (-3,0)$ and Γ_0''' is the lower left elliptic arc connecting points $A_2 = (-3,0)$ and $E_2 = (-1,-2)$. Let us consider the intersection points of the hyperbolic characteristics:
$\Gamma_1 \cap \Gamma_1' = \{P_1\}$, where $P_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$; $\Gamma_2 \cap \Gamma_2' = \{P_2\}$, where $P_2 = \left(\frac{3}{2}, \frac{1}{2}\right)$;
$\Delta_1 \cap \Delta_1' = \{P_1'\}$, where $P_1' = \left(\frac{-5}{2}, \frac{1}{2}\right)$; $\Delta_2 \cap \Delta_2' = \{P_2'\}$, where $P_2' = \left(\frac{-3}{2}, \frac{1}{2}\right)$;
$\gamma_1 \cap \gamma_1' = \{Q_1\}$, where $Q_1 = \left(\frac{-1}{2}, \frac{3}{2}\right)$; $\gamma_2 \cap \gamma_2' = \{Q_2\}$, where $Q_2 = \left(\frac{-1}{2}, \frac{5}{2}\right)$;
$\delta_1 \cap \delta_1' = \{Q_1'\}$, where $Q_1' = \left(\frac{-1}{2}, \frac{-3}{2}\right)$; $\delta_2 \cap \delta_2' = \{Q_2'\}$, where $Q_2' = \left(\frac{-1}{2}, \frac{-1}{2}\right)$.

Note that:

1. The boundary ∂D is assumed to be a piecewise continuously differentiable arc. The elliptic arcs are "star-shaped" (counterclockwise).

2. We consider continuous solutions u of the quaterelliptic -quaterhyperbolic equation (*) with eight parabolic lines, which have the property that u_x , u_y are continuous in the closure $\bar{D} = D \cup \partial D$.

These continuity conditions may be weakened at the eight "singular" points
$A_1, A_2, B_1, B_2, O_1, O_2, O_1', O_2'$, by considering u_x , u_y continuous on the boundary ∂D except at these points. By "quaterelliptic" and "quaterhyperbolic" we mean that Bitsadze-Lavrentjev equation (1.2) is elliptic in four different subdomains and hyperbolic in four other "symmetric" subdomains of the whole domain D.

In fact, equation (1.2) is elliptic and hyperbolic in $G_1 \cup G_1' \cup G_1'' \cup G_1'''$ and $G_2 \cup G_2' \cup G_2'' \cup G_2'''$, respectively.

Definition 2.1. A function $u = u(x,y)$ is a quasi-regular solution ([7], [8], [10]-[16]) of Problem (BL) if

    i) $u \in C^2(D) \cap C(\overline{D})$, $\quad \overline{D} = D \cup \partial D$;

ii) the Green's theorem (of the integral calculus) is applicable to the integrals

$$\iint_D u_x L u \, dx dy, \quad \iint_D u_y L u \, dx dy \quad ;$$

iii) the boundary and region integrals, which arise, exist; and

iv) u satisfies the Bitsadze-Lavrentjev equation (1.2) in D and the following boundary condition on the exterior boundary $Ext(D)$:

$$u = \begin{cases} \varphi_1(s) & on & \Gamma_0 & ; & \varphi_2(s) & on & \Gamma_0' \\ \varphi_3(s) & on & \Gamma_0'' & ; & \varphi_4(s) & on & \Gamma_0''' \\ \psi_1(x) & on & \Gamma_2 & ; & \psi_2(x) & on & \Gamma_2' \\ \psi_3(x) & on & \gamma_2 & ; & \psi_4(x) & on & \gamma_2' \\ \psi_5(x) & on & \Delta_1 & ; & \psi_6(x) & on & \Delta_1' \\ \psi_7(x) & on & \delta_1 & ; & \psi_8(x) & on & \delta_1' \end{cases} , \tag{2.1}$$

with continuous prescribed values.

The exterior Bitsadze-Lavrentjev problem: Tricomi Case or Problem (BL-T):

consists of finding a solution u of the quaterelliptic -quaterhyperbolic equation (1.2) with eight parabolic lines in D and which assumes continuous prescribed values (2.1).

Theorem 2.2. (UNIQUENESS THEOREM): Consider the general Bitsadze-Lavrentjev quaterelliptic - quaterhyperbolic equation (1.2) with eight parabolic lines and the boundary condition (2.1). Assume the above mixed doubly connected domain D and the following conditions:

(R_1) $r \leq 0$ on $Int(D) = (\Gamma_1 \cup \Gamma_1') \cup (\gamma_1 \cup \gamma_1') \cup (\Delta_2 \cup \Delta_2') \cup (\delta_2 \cup \delta_2')$,

(R_2) $\begin{cases} xdy - (y-1)dx \geq 0 & on & \Gamma_0 & ; & xdy - ydx \geq 0 & on & \Gamma_0' \\ (x+1)dy - (y-1)dx \geq 0 & on & \Gamma_0'' & ; & (x+1)dy - ydx \geq 0 & on & \Gamma_0''' \end{cases}$,

(R_3) $\begin{cases} 2r + xr_x + (y-1)r_y \leq 0 & in & G_1 & ; & r + xr_x \leq 0 & in & G_2 \\ 2r + xr_x + yr_y \leq 0 & in & G_1' & ; & r + (y-1)r_y \leq 0 & in & G_2' \\ 2r + (x+1)r_x + (y-1)r_y \leq 0 & in & G_1'' & ; & r + (x+1)r_x \leq 0 & in & G_2'' \\ 2r + (x+1)r_x + yr_y \leq 0 & in & G_1''' & ; & r + yr_y \leq 0 & in & G_2''' \end{cases}$,

(R_4) $K = M = +1$, in $G_1 \cup G_1' \cup G_1'' \cup G_1'''$,

(R_5) $\begin{cases} K = -1 & , & M = +1 & in & G_2 \cup G_2'' \\ K = +1 & , & M = -1 & in & G_2' \cup G_2''' \end{cases}$.

Let $()_x = \partial()/\partial x$, $()^\bullet = d()/dx$, $()_y = \partial()/\partial y$, $()' = d()/dy$,
where $f = f(x, y)$ is continuous in D, $r = r(x, y)$ is once-continuously differentiable in D, $K = K(y)) := sgn(y(y-1)|k(y)|)$ is a discontinuous function of y, for $y \in [-k_1, k_2]$, with $-k_1 = inf\{y : (x, y) \in D\}$ and $k_2 = sup\{y : (x, y) \in D\}$, and $M = M(x) := sgn(x(x+1)|m(x)|)$ is discontinuous for $x \in [-m_1, m_2]$ with $-m_1 = inf\{x : (x, y) \in D\}$ and $m_2 = sup\{x : (x, y) \in D\}$, such that $k = k(y) \neq 0$; $m = m(x) \neq 0$,everywhere in D, as well as

$$K = \begin{cases} +1 & for & \{y < 0\} \cup \{y > 1\} \\ 0 & for & \{y = 0\} \cup \{y = 1\} \\ -1 & for & \{0 < y < 1\} \end{cases} ; \quad M = \begin{cases} +1 & for & \{x < -1\} \cup \{x > 0\} \\ 0 & for & \{x = 0\} \cup \{x = -1\} \\ -1 & for & \{-1 < x < 0\} \end{cases} .$$

Then Problem (BL-T) has at most one quasi-regular solution in D.

Proof. We apply the well-known $a - b - c$ energy integral method with $a = 0$, and use the above Bitsadze-Lavrentjev equation (1.2) and the boundary condition (2.1).

First, we assume two quasi-regular solutions u_1, u_2 of Problem (BL-T). Then we claim that $u = u_1 - u_2 = 0$ holds in the domain D. In fact, we investigate

$$0 = J = 2 \langle lu, Lu \rangle_0 = \iint_D 2luLu \, dxdy, \qquad (2.2)$$

where $lu = b(x) u_x + c(y) u_y$, and $Lu = L(u_1 - u_2) = Lu_1 - Lu_2 = f - f = 0$ in D, with choices

$$b = b(x) = \begin{cases} x & in & G_1 \cup G_1' \cup G_2 \\ x+1 & in & G_1'' \cup G_1''' \cup G_2'' \\ 0 & in & G_2' \cup G_2''' \end{cases}, \quad c = c(y) = \begin{cases} y & in & G_1' \cup G_1''' \cup G_2''' \\ y-1 & in & G_1 \cup G_1'' \cup G_2' \\ 0 & in & G_2 \cup G_2'' \end{cases}. \quad (2.3)$$

We consider the new differential identities

$$2bu_x u_{xx} = \left(bu_x^2\right)_x - (b)^\bullet u_x^2; \quad 2bu_x u_{yy} = (2bu_x u_y)_y - \left(bu_y^2\right)_x + (b)^\bullet u_y^2,$$

$$2cu_y u_{xx} = (2cu_x u_y)_x - \left(cu_x^2\right)_y + (c)' u_x^2; \quad 2cu_y u_{yy} = \left(cu_y^2\right)_y - (c)' u_y^2,$$

$$2bruu_x = \left(bru^2\right)_x - (br)_x u^2; \quad 2cruu_y = \left(cru^2\right)_y - (cr)_y u^2.$$

Employing these identities and the classical Green's theorem of the integral calculus separately in G_i^j $(i = 1, 2; j = 0, 1, 2, 3)$, due to the discontinuity of K, M, we obtain from (1.2), (2.2)-(2.3) that

$$0 = J_{G_i^j} = I_{G_i^j} + I_{\partial G_i^j} = \int_{G_i^j} 2 \left[bu_x + cu_y\right] \left[K(y)u_{xx} + M(x)u_{yy} + ru\right] dxdy, \qquad (2.4)$$

where, if we denote:

$$G_1 = G_1^0, G_1' = G_1^1, \ G_1'' = G_1^2, \ G_1''' = G_1^3 \quad ; \quad G_2 = G_2^0, \ G_2' = G_2^1, \ G_2'' = G_2^2, \ G_2''' = G_2^3 \quad ,$$

then

$$I_{G_i^j} = \int_{G_i^j} Q(u_x, u_y) \, dxdy = \int_{G_i^j} \left(Au_x^2 + Bu_y^2 + \Gamma u^2 + 2\Delta u_x u_y\right) dxdy;$$

$$I_{\partial G_i^j} = \int_{\partial G_i^j} \tilde{Q}(u_x, u_y) ds = \int_{\partial G_i^j} \left(\tilde{A}u_x^2 + \tilde{B}u_y^2 + \tilde{\Gamma}u^2 + 2\tilde{\Delta}u_x u_y\right) ds,$$

with

$$Q = Q(u_x, u_y) = Au_x^2 + Bu_y^2 + \Gamma u^2 + 2\Delta u_x u_y,$$

where

$$A = \left[K\left(-\dot{b} + c'\right)\right], \ \ B = M\left[\dot{b} - c'\right], \ \ \Gamma = -\left[(br)_x + (cr)_y\right], \ \ \Delta = 0,$$

in G_i^j, and

$$\tilde{Q} = \tilde{Q}(u_x, u_y) = \tilde{A}u_x^2 + \tilde{B}u_y^2 + \tilde{\Gamma}u^2 + 2\tilde{\Delta}u_x u_y,$$

where

$$\tilde{A} = (bv_1 - cv_2)K, \ \tilde{B} = (-bv_1 + cv_2)M, \ \tilde{\Gamma} = (bv_1 + cv_2)r, \tilde{\Delta} = bMv_2 + cKv_1,$$

on ∂G_i^j, where $v = (v_1, v_2) = (dy/ds, -dx/ds)$ is the outer unit normal vector on the boundary ∂G_i^j, of the domain G_i^j, with $0 < ds = \sqrt{(dx)^2 + (dy)^2}$, $|\nu| = 1$, and

$$\iint\limits_{G_i^j} ()_x \, dxdy = \int\limits_{\partial G_i^j} () \, v_1 ds, \quad \iint\limits_{G_i^j} ()_y \, dxdy = \int\limits_{\partial G_i^j} () \, v_2 ds,$$

are the Green's integral formulas.

From the above conditions of the uniqueness theorem 2.2, we obtain

$$A = B = \begin{cases} 1 & in & G_2 \cup G_2' \cup G_2'' \cup G_2''' \\ 0 & in & G_1 \cup G_1' \cup G_1'' \cup G_1''' \end{cases} \quad .$$

Also

$$0 \leq \Gamma = \begin{cases} -(2r + xr_x + (y-1)r_y) & in & G_1 \\ -(2r + xr_x + yr_y) & in & G_1' \\ -(2r + (x+1)r_x + (y-1)r_y) & in & G_1'' \\ -(2r + (x+1)r_x + yr_y) & in & G_1''' \end{cases} := \begin{cases} -(r + xr_x) & in & G_2 \\ -(r + (y-1)r_y) & in & G_2' \\ -(r + (x+1)r_x) & in & G_2'' \\ -(r + yr_y) & in & G_2''' \end{cases} \quad .$$

We note:

1. In the upper right elliptic region $\bar{G}_1 = G_1 \cup \partial G_1 : 0 = J_{G_1} = I_{G_1} + I_{\partial G_1}$:

$$I_{G_1} = - \iint\limits_{G_1} (2r + xr_x + (y-1)r_y) u^2 dxdy \geq 0;$$

$$I_{\partial G_1} = \int\limits_{\Gamma_0} (x\nu_1 + (y-1)\nu_2)(\nu_1^2 + \nu_2^2) N^2 ds$$

$$+ 2 \int\limits_{Z_1 O_1} (y-1)\nu_1 u_x u_y ds + 2 \int\limits_{O_1 B_1} x\nu_2 u_x u_y ds ;$$

$u|_{\Gamma_0} = 0$ *yields* $u_x = N\nu_1$; $u_y = N\nu_2$, $N = normalizing \ factor$, implying:

$$\begin{aligned} Q_0 &= Q_0(u_x, u_y) := (x\nu_1 - (y-1)\nu_2) u_x^2 + (-x\nu_1 + (y-1)\nu_2) u_y^2 \\ &+ 2(x\nu_2 + (y-1)\nu_1) u_x u_y = (x\nu_1 + (y-1)\nu_2)(\nu_1^2 + \nu_2^2) N^2 > 0 \end{aligned} \quad .$$

Note that $K = M = 1$ and $b = x$, $c = y - 1$ in G_1, as well as $u = 0$ on Γ_0.

2. In the right hyperbolic region $\bar{G}_2 = G_2 \cup \partial G_2 : 0 = J_{G_2} = I_{G_2} + I_{\partial G_2}$:

$$I_{G_2} = \iint\limits_{G_2} \left[u_x^2 + u_y^2 - (r + x r_x) u^2 \right] dx dy \geq 0;$$

$$I_{\partial G_2} = \int\limits_{\Gamma_1} x \left[(u_x - u_y)^2 - r u^2 \right] dx + \int\limits_{\Gamma_1'} (-x) \left[(u_x + u_y)^2 - r u^2 \right] dx$$

$$+ 2 \int\limits_{B_1 O_1} x \nu_2 u_x u_y ds + 2 \int\limits_{O_2 B_2} x \nu_2 u_x u_y ds;$$

as

$$I_{\Gamma_1} = \int\limits_{\Gamma_1} x \left[-\nu_1 \left(u_x^2 + u_y^2 \right) + 2\nu_2 u_x u_y + \nu_1 r u^2 \right] ds$$

$$= - \int\limits_{\Gamma_1} x \left[\left(u_x^2 + u_y^2 \right) dy + 2 u_x u_y dx - r u^2 dy \right]$$

$$= \int\limits_{\Gamma_1} x \left[(u_x - u_y)^2 - r u^2 \right] dx \geq 0,$$

where on $\Gamma_1 : y = -x + 1, \ or \ dy = -dx \ : \ \nu_1 = \nu_2$, and $x dx|_{\Gamma_1} = -x\nu_2 ds|_{\Gamma_1} > 0$, as well as

$$I_{\Gamma_1'} = \int\limits_{\Gamma_1'} x \left[-\nu_1 \left(u_x^2 + u_y^2 \right) + 2\nu_2 u_x u_y + \nu_1 r u^2 \right] ds$$

$$= - \int\limits_{\Gamma_1'} x \left[\left(u_x^2 + u_y^2 \right) dy + 2 u_x u_y dx - r u^2 dy \right]$$

$$= \int\limits_{\Gamma_1'} (-x) \left[(u_x + u_y)^2 - r u^2 \right] dx \geq 0,$$

where on $\Gamma_1' : y = x, \ or \ dy = dx \ : \ \nu_1 = -\nu_2$, and $(-x) dx|_{\Gamma_1'} = x\nu_2 ds|_{\Gamma_1'} > 0$, as well as

$$I_{\Gamma_2} = \int\limits_{\Gamma_2} (-x) \left[(u_x + u_y)^2 - r u^2 \right] dx = 0;$$

and

$$I_{\Gamma_2'} = \int\limits_{\Gamma_2'} x \left[(u_x - u_y)^2 - r u^2 \right] dx = 0 :$$

$u|_{\Gamma_2 \cup \Gamma_2'} = 0 \ yields \ u_x = N\nu_1 \ ; \ u_y = N\nu_2, N = normalizing \ factor$, implying: $(u_x + u_y)|_{\Gamma_2} = [(\nu_1 + \nu_2) N]|_{\Gamma_2} = \left[\left(\frac{dy - dx}{ds} \right) N \right]|_{\Gamma_2} = 0$, as on $\Gamma_2 : y = x - 1, \ or \ dy = dx$, and

$(u_x - u_y)|_{\Gamma_2'} = [(\nu_1 - \nu_2) N]|_{\Gamma_2'} = \left[\left(\frac{dy+dx}{ds}\right) N\right]|_{\Gamma_2'} = 0$, as on $\Gamma_2' : y = -x + 2$, or $dy = -dx$; $I_{O_1B_1} + I_{B_1O_1} = 0$, because $\nu_2|_{O_1B_1} = -\nu_2|_{B_1O_1}$.

Note that $K = -1$, $M = 1$ and $b = x$, $c = 0$ in G_2, as well as $u = 0$ on $\Gamma_2 \cup \Gamma_2'$.

3. In the lower right elliptic region $\bar{G}_1' = G_1' \cup \partial G_1' : 0 = J_{G_1'} = I_{G_1'} + I_{\partial G_1'}$:

$$I_{G_1'} = -\iint\limits_{G_1'} (2r + xr_x + yr_y) u^2 dxdy \geq 0;$$

$$I_{\partial G_1'} = \int\limits_{\Gamma_0'} (x\nu_1 + y\nu_2) \left(\nu_1^2 + \nu_2^2\right) N^2 ds + 2 \int\limits_{B_2O_2} x\nu_2 u_x u_y ds + 2 \int\limits_{O_2Z_2} y\nu_1 u_x u_y ds \ ;$$

$u|_{\Gamma_0'} = 0$ $yields$ $u_x = N\nu_1$; $u_y = N\nu_2, N = normalizing$ $factor$, implying:

$$\begin{aligned} Q_1 &= Q_1\left(u_x, u_y\right): \\ &= \left(x\nu_1 - y\nu_2\right)\left(u_x^2 - u_y^2\right) + 2\left(x\nu_2 + y\nu_1\right) u_x u_y \\ &= \left(x\nu_1 + y\nu_2\right)\left(\nu_1^2 + \nu_2^2\right) N^2 > 0; \end{aligned}$$

$I_{O_2B_2} + I_{B_2O_2} = 0$, because $\nu_2|_{O_2B_2} = -\nu_2|_{B_2O_2}$. Note that $K = M = 1$ and $b = x$, $c = y$ in G_1, as well as $u = 0$ on Γ_0'.

4. In the upper hyperbolic region $\bar{G}_2' = G_2' \cup \partial G_2' : 0 = J_{G_2'} = I_{G_2'} + I_{\partial G_2'}$:

$$I_{G_2'} = \iint\limits_{G_2'} \left[u_x^2 + u_y^2 - (r + (y - 1) r_y) u^2\right] dxdy \geq 0;$$

$$\begin{aligned} I_{\partial G_2'} &= \int\limits_{\gamma_1} (y - 1)\left[(u_x - u_y)^2 - ru^2\right]dx + \int\limits_{\gamma_1'} (y - 1)\left[(u_x + u_y)^2 - ru^2\right]dx \\ &\quad + 2 \int\limits_{O_1Z_1} (y-1)\nu_1 u_x u_y ds + 2 \int\limits_{E_1O_1'} (y-1)\nu_1 u_x u_y ds \ ; \end{aligned}$$

as

$$\begin{aligned} I_{\gamma_1} &= \int\limits_{\gamma_1} (y - 1)\left[-\nu_2\left(u_x^2 + u_y^2\right) + 2\nu_1 u_x u_y + \nu_2 ru^2\right]ds \\ &= \int\limits_{\gamma_1} (y - 1)\left[\left(u_x^2 + u_y^2\right) dx + 2u_x u_y dy - ru^2 dx\right] \\ &= \int\limits_{\gamma_1} (y - 1)\left[(u_x - u_y)^2 - ru^2\right]dx \geq 0, \end{aligned}$$

where on γ_1 (: *extension of* Γ_1) : $y = -x + 1$, or $dy = -dx$: $\nu_1 = \nu_2$, and $(y-1)\,dx|_{\gamma_1} = -(y-1)\,\nu_2 ds|_{\gamma_1} > 0$, as well as

$$I_{\gamma_1'} = \int_{\gamma_1'} (y-1)\left[-\nu_2\left(u_x^2 + u_y^2\right) + 2\nu_1 u_x u_y + \nu_2 r u^2\right] ds$$

$$= \int_{\gamma_1'} (y-1)\left[\left(u_x^2 + u_y^2\right) dx + 2u_x u_y dy - r u^2 dx\right]$$

$$= \int_{\gamma_1'} (y-1)\left[\left(u_x + u_y\right)^2 - r u^2\right] dx \geq 0,$$

where on γ_1' : $y = x + 2$, or $dy = dx$: $\nu_1 = -\nu_2$, and $(y-1)\,dx|_{\gamma_1'} = -(y-1)\,\nu_2 ds|_{\gamma_1'} > 0$, as well as

$$I_{\gamma_2} = \int_{\gamma_2} (y-1)\left[\left(u_x + u_y\right)^2 - r u^2\right] dx = 0;$$

and

$$I_{\gamma_2'} = \int_{\gamma_2'} (y-1)\left[\left(u_x - u_y\right)^2 - r u^2\right] dx = 0 :$$

$u|_{\gamma_2 \cup \gamma_2'} = 0$ *yields* $u_x = N\nu_1$; $u_y = N\nu_2$, $N = normalizing\ factor$ implying: $(u_x + u_y)|_{\gamma_2} = [(\nu_1 + \nu_2)N]|_{\gamma_2} = \left[\left(\frac{dy-dx}{ds}\right)N\right]\Big|_{\gamma_2} = 0$, as on $\gamma_2 : y = x+3$, or $dy = dx$, and $(u_x - u_y)|_{\gamma_2'} = [(\nu_1 - \nu_2)N]|_{\gamma_2'} = \left[\left(\frac{dy+dx}{ds}\right)N\right]\Big|_{\gamma_2'} = 0$, as on $\gamma_2' : y = -x+2$, or $dy = -dx$; $I_{Z_1 O_1} + I_{O_1 Z_1} = 0$, because $\nu_2|_{Z_1 O_1} = -\nu_2|_{O_1 Z_1}$. Note that $K = 1$, $M = -1$ and $b = 0$, $c = y - 1$ in G_2', as well as $u = 0$ on $\gamma_2 \cup \gamma_2'$.

5. In the upper left elliptic region $\bar{G}_1'' = G_1'' \cup \partial G_1'' : 0 = J_{G_1''} = I_{G_1''} + I_{\partial G_1''}$:

$$I_{G_1''} = -\iint_{G_1''} \left(2r + (x+1)r_x + (y-1)r_y\right)u^2 dx dy \geq 0;$$

$$I_{\partial G_1''} = \int_{\Gamma_0''} ((x+1)\nu_1 + (y-1)\nu_2)\left(\nu_1^2 + \nu_2^2\right)N^2 ds$$

$$+ 2\int_{A_1 O_1'} (x+1)\nu_2 u_x u_y ds + 2\int_{O_1' E_1} (y-1)\nu_1 u_x u_y ds;$$

$u|_{\Gamma_0''} = 0$ *yields relations*

$$u_x = N\nu_1 \ ; \ u_y = N\nu_2,$$

$N = normalizing\ factor$, implying:

$$\begin{aligned}
Q_2 &= Q_2\left(u_x, u_y\right) : = \ ((x+1)\nu_1 - (y-1)\nu_2)\left(u_x^2 - u_y^2\right) \\
&\quad + 2\left((x+1)\nu_2 + (y-1)\nu_1\right)u_x u_y \\
&= ((x+1)\nu_1 + (y-1)\nu_2)\left(\nu_1^2 + \nu_2^2\right)N^2 > 0;
\end{aligned}$$

$I_{E_1 O_1'} + I_{O_1' E_1} = 0$, because $\nu_1|_{E_1 O_1'} = -\nu_1|_{O_1' E_1}$. $I_{O_2 B_2} + I_{B_2 O_2} = 0$, because $\nu_2|_{O_2 B_2} = -\nu_2|_{B_2 O_2}$. Note that $K = M = 1$ and $b = x + 1$, $c = y - 1$ in G_1'', as well as $u = 0$ on Γ_0''.

6. In the left hyperbolic region: $\bar{G}_2'' = G_2'' \cup \partial G_2'' : 0 = J_{G_2''} = I_{G_2''} + I_{\partial G_2''} :$

$$I_{G_2''} = \iint\limits_{G_2''} \left[u_x^2 + u_y^2 - (r + (x + 1)\, r_x)\, u^2 \right] dx dy \geq 0;$$

$$I_{\partial G_2''} = \int\limits_{\Delta_2'} (x + 1) \left[(u_x - u_y)^2 - ru^2 \right] dx + \int\limits_{\Delta_2} (-(x + 1)) \left[(u_x + u_y)^2 - ru^2 \right] dx$$

$$+ 2 \int\limits_{O_1' A_1} (x + 1)\, \nu_2 u_x u_y ds + 2 \int\limits_{A_2 O_2'} (x + 1)\, \nu_2 u_x u_y ds;$$

as

$$I_{\Delta_2} = \int\limits_{\Gamma_1} (x + 1) \left[-\nu_1 \left(u_x^2 + u_y^2 \right) + 2\nu_2 u_x u_y + \nu_1 ru^2 \right] ds$$

$$= - \int\limits_{\Delta_2} (x + 1) \left[\left(u_x^2 + u_y^2 \right) dy + 2 u_x u_y dx - ru^2 dy \right]$$

$$= - \int\limits_{\Delta_2} (x + 1) \left[(u_x + u_y)^2 - ru^2 \right] dx \geq 0,$$

where on $\Delta_2 : y = x + 2$, or $dy = dx$: $\nu_1 = -\nu_2$, and $-(x + 1)\, dx|_{\Delta_2} = (x + 1)\, \nu_2 ds|_{\Delta_2} > 0$, as well as

$$I_{\Delta_2'} = \int\limits_{\Delta_2'} (x + 1) \left[-\nu_1 \left(u_x^2 + u_y^2 \right) + 2\nu_2 u_x u_y + \nu_1 ru^2 \right] ds$$

$$= - \int\limits_{\Delta_2'} (x + 1) \left[\left(u_x^2 + u_y^2 \right) dy + 2 u_x u_y dx - ru^2 dy \right]$$

$$= \int\limits_{\Delta_2'} (x + 1) \left[(u_x - u_y)^2 - ru^2 \right] dx \geq 0,$$

where on $\Delta_2' : y = -x - 1$, or $dy = -dx$: $\nu_1 = \nu_2$, and $(x + 1)\, dx|_{\Delta_2'} = -(x + 1)\, \nu_2 ds|_{\Delta_2'} > 0$, as well as

$$I_{\Delta_1'} = \int\limits_{\Delta_1'} (-(x + 1)) \left[(u_x + u_y)^2 - ru^2 \right] dx = 0;$$

and

$$I_{\Delta_1} = \int\limits_{\Delta_1} (x + 1) \left[(u_x - u_y)^2 - ru^2 \right] dx = 0 :$$

$u|_{\Delta_1 \cup \Delta_1'} = 0$ *yields* , $u_x = N\nu_1$; $u_y = N\nu_2$, $N = $ *normalizing factor* implying:
$(u_x + u_y)|_{\Delta_1'} = [(\nu_1 + \nu_2) N]|_{\Delta_1'} = \left[\left(\frac{dy-dx}{ds}\right) N\right]|_{\Delta_1'} = 0$, as on $\Delta_1' : y = x + 3$, or $dy = dx$,
and $(u_x - u_y)|_{\Delta_1} = [(\nu_1 - \nu_2) N]|_{\Delta_1} = \left[\left(\frac{dy+dx}{ds}\right) N\right]|_{\Delta_1} = 0$, as on $\Delta_1 : y = -x -$
2, or $dy = -dx$; $I_{A_1O_1'} + I_{O_1'A_1} = 0$, because $\nu_2|_{A_1O_1'} = -\nu_2|_{O_1'A_1}$. Note that $K = -1$, $M = 1$
and $b = x + 1$, $c = 0$ in G_2'', as well as $u = 0$ on $\Delta_1 \cup \Delta_1'$.

7. In the lower elliptic region: $\bar{G}_1''' = G_1''' \cup \partial G_1''' : 0 = J_{G_1'''} = I_{G_1'''} + I_{\partial G_1'''}$:

$$I_{G_1'''} = -\iint\limits_{G_1'''} (2r + (x + 1) r_x + yr_y) u^2 dxdy \geq 0;$$

$$I_{\partial G_1'''} = \int\limits_{\Gamma_0'''} ((x + 1) \nu_1 + y\nu_2) \left(\nu_1^2 + \nu_2^2\right) N^2 ds$$

$$+ 2 \int\limits_{E_2O_2'} y\nu_1 u_x u_y ds + 2 \int\limits_{O_2'A_2} (x + 1) \nu_2 u_x u_y ds;$$

$u|_{\Gamma_0'''} = 0$ *yields* $u_x = N\nu_1$; $u_y = N\nu_2$, $N = normalizing\ factor$, implying:

$$Q_3 = Q_3 (u_x, u_y) :$$
$$= ((x + 1) \nu_1 - y\nu_2) \left(u_x^2 - u_y^2\right) + 2 ((x + 1) \nu_2 + y\nu_1) u_x u_y$$
$$= ((x + 1) \nu_1 + y\nu_2) \left(\nu_1^2 + \nu_2^2\right) N^2 > 0;$$

$I_{A_2O_2'} + I_{O_2'A_2} = 0$, because $\nu_2|_{A_2O_2'} = -\nu_2|_{O_2'A_2}$.
Note that $K = M = 1$ and $b = x + 1$, $c = y$ in G_1''', as well as $u = 0$ on Γ_0'''.

8. In the lower hyperbolic region: $\bar{G}_2''' = G_2''' \cup \partial G_2''' : 0 = J_{G_2'''} = I_{G_2'''} + I_{\partial G_2'''}$:

$$I_{G_2'''} = \iint\limits_{G_2'''} \left[u_x^2 + u_y^2 - (r + yr_y) u^2\right] dxdy \geq 0;$$

$$I_{\partial G_2'''} = \int\limits_{\delta_2'} y \left[(u_x - u_y)^2 - ru^2\right] dx + \int\limits_{\delta_2} y \left[(u_x + u_y)^2 - ru^2\right] dx$$

$$+ 2 \int\limits_{Z_2O_2} y\nu_1 u_x u_y ds + 2 \int\limits_{O_2'E_2} y\nu_1 u_x u_y ds;$$

as

$$I_{\delta_2} = \int\limits_{\delta_2} y \left[-\nu_2 \left(u_x^2 + u_y^2\right) + 2\nu_1 u_x u_y + \nu_2 ru^2\right] ds$$

$$= \int\limits_{\delta_2} y \left[\left(u_x^2 + u_y^2\right) dx + 2u_x u_y dy - ru^2 dx\right] = \int\limits_{\delta_2} y \left[(u_x + u_y)^2 - ru^2\right] dx \geq 0,$$

where on δ_2 $(: extension\ of\ \Gamma_1') : y = x$, $or\ dy = dx$ $:$ $\nu_1 = -\nu_2$, and $ydx|_{\delta_2} = -y\nu_2 ds|_{\delta_2} > 0$, as well as

$$I_{\delta_2'} = \int_{\delta_2'} y\left[-\nu_2\left(u_x^2 + u_y^2\right) + 2\nu_1 u_x u_y + \nu_2 r u^2\right]ds = \int_{\delta_2'} y\left[\left(u_x^2 + u_y^2\right)dx + 2u_x u_y dy - ru^2 dx\right]$$

$$= \int_{\delta_2'} y\left[\left(u_x - u_y\right)^2 - ru^2\right]dx \geq 0,$$

where on δ_2' $(extension\ of\ \Delta_2') : y = -x - 1$, $or\ dy = -dx$ $:$ $\nu_1 = \nu_2$, and $ydx|_{\delta_2'} = -y\nu_2 ds|_{\delta_2'} > 0$, as well as $I_{\delta_1} = \int_{\delta_1} y\left[\left(u_x - u_y\right)^2 - ru^2\right]dx = 0$;

and $I_{\delta_1'} = \int_{\delta_1'} y\left[\left(u_x + u_y\right)^2 - ru^2\right]dx = 0$:

$u|_{\delta_1 \cup \delta_1'} = 0\ yields\ u_x = N\nu_1$; $u_y = N\nu_2$, $N = normalizing\ factor$, implying: $(u_x + u_y)|_{\delta_1'} = [(\nu_1 + \nu_2)N]|_{\delta_1'} = \left[\left(\frac{dy-dx}{ds}\right)N\right]|_{\delta_1'} = 0$, as on $\delta_1' : y = x-1$, $or\ dy = dx$, and $(u_x - u_y)|_{\delta_1} = [(\nu_1 - \nu_2)N]|_{\delta_1} = \left[\left(\frac{dy+dx}{ds}\right)N\right]|_{\delta_1} = 0$, as on $\delta_1 : y = -x-2$, $or\ dy = -dx$; $I_{Z_2 O_2} + I_{O_2 Z_2} = 0$, because $\nu_1|_{Z_2 O_2} = -\nu_1|_{O_2 Z_2}$. $I_{E_2 O_2'} + I_{O_2' E_2} = 0$, because $\nu_1|_{E_2 O_2'} = -\nu_1|_{O_2' E_2}$. Note : $K = 1$, $M = -1$ and $b = 0$, $c = y$ in G_2''', and $u = 0$ on $\delta_1 \cup \delta_1'$. On the eight parabolic segments, the following "vanishing" relation holds:

$$\left(I_{O_1 B_1} + I_{B_1 O_1}\right) + \left(I_{O_2 B_2} + I_{B_2 O_2}\right) + \left(I_{O_1 Z_1} + I_{Z_1 O_1}\right)$$
$$+ \left(I_{O_2 Z_2} + I_{Z_2 O_2}\right) + \left(I_{O_1' A_1} + I_{A_1 O_1'}\right) + \left(I_{O_2' A_2} + I_{A_2 O_2'}\right)$$
$$+ \left(I_{O_1' E_1} + I_{E_1 O_1'}\right) + \left(I_{O_1' E_2} + I_{E_2 O_2'}\right) = 0.$$

Also on the four pairs of characteristics: Γ_2, Γ_2' ; γ_2, γ_2' ; Δ_1, Δ_1' ; δ_1, δ_1' :

$$\left(I_{\Gamma_2} + I_{\Gamma_2'}\right) + \left(I_{\gamma_2} + I_{\gamma_2'}\right) + \left(I_{\Delta_1} + I_{\Delta_1'}\right) + \left(I_{\delta_1} + I_{\delta_1'}\right) = 0.$$

Applying the energy integral method in each of the eight regions:
G_i , G_i' , G_i'' , G_i''' , $i = 1, 2$, separately, due to the discontinuity of

$$K = \begin{cases} +1 & for & \{y < 0\} \cup \{y > 1\} \\ 0 & for & \{y = 0\} \cup \{y = 1\} \\ -1 & for & \{0 < y < 1\} \end{cases} \quad ; \quad M = \begin{cases} +1 & for & \{x < -1\} \cup \{x > 0\} \\ 0 & for & \{x = 0\} \cup \{x = -1\} \\ -1 & for & \{-1 < x < 0\} \end{cases} ,$$

in D, Green's theorem cannot be applied directly in the whole mixed domain D, and then adding all the pertinent integral expressions, we find $0 = J_D = I_D + I_{\partial D} \geq 0$, where $I_D \geq$

0 ; $I_{\partial D} \geq 0$. In fact,

$$I_D = -\iint\limits_{G_1} (2r + xr_x + (y-1)\,r_y)\,u^2 dxdy - \iint\limits_{G_1'} (2r + xr_x + yr_y)\,u^2 dxdy$$

$$- \iint\limits_{G_1''} (2r + (x+1)\,r_x + (y-1)\,r_y)\,u^2 dxdy$$

$$- \iint\limits_{G_1'''} (2r + (x+1)\,r_x + yr_y)\,u^2 dxdy + \iint\limits_{G_2 \cup G_2' \cup G_2'' \cup G_2'''} (u_x^2 + u_y^2)\,dxdy$$

$$- \iint\limits_{G_2} (r + xr_x)\,u^2 dxdy - \iint\limits_{G_2'} (r + (y-1)\,r_y)\,u^2 dxdy$$

$$- \iint\limits_{G_2''} (r + (x+1)\,r_x)\,u^2 dxdy - \iint\limits_{G_2'''} (r + yr_y)\,u^2 dxdy \;\geq\; 0.$$

Also

$$I_{\partial D} = \int\limits_{\Gamma_0} (x\nu_1 + (y-1)\,\nu_2)\,(\nu_1^2 + \nu_2^2)\,N^2 ds + \int\limits_{\Gamma_0'} (x\nu_1 + y\nu_2)\,(\nu_1^2 + \nu_2^2)\,N^2 ds$$

$$+ \int\limits_{\Gamma_0''} ((x+1)\,\nu_1 + (y-1)\,\nu_2)\,(\nu_1^2 + \nu_2^2)\,N^2 ds$$

$$+ \int\limits_{\Gamma_0'''} ((x+1)\,\nu_1 + y\nu_2)\,(\nu_1^2 + \nu_2^2)\,N^2 ds + \int\limits_{\Gamma_1} x\left[(u_x - u_y)^2 - ru^2\right] dx$$

$$- \int\limits_{\Gamma_1'} x\left[(u_x + u_y)^2 - ru^2\right] dx + \int\limits_{\gamma_1} (y-1)\left[(u_x - u_y)^2 - ru^2\right] dx$$

$$+ \int\limits_{\gamma_1'} (y-1)\left[(u_x + u_y)^2 - ru^2\right] dx - \int\limits_{\Delta_2} (x+1)\left[(u_x + u_y)^2 - ru^2\right] dx$$

$$+ \int\limits_{\Delta_2'} (x+1)\left[(u_x - u_y)^2 - ru^2\right] dx + \int\limits_{\delta_2} y\left[(u_x + u_y)^2 - ru^2\right] dx$$

$$+ \int\limits_{\delta_2'} y\left[(u_x - u_y)^2 - ru^2\right] dx \;\geq\; 0.$$

From the maximum principle and the uniqueness of the solution of the Cauchy problem on hyperbolic equations, one obtains $u = u(x,y) \equiv 0$, everywhere in the whole mixed domain D, completing the proof of our uniqueness theorem. In fact, from the maximum principle, $u|_{\bar{G}_2 \cup \bar{G}_2' \cup \bar{G}_2'' \cup \bar{G}_2'''} = 0$, implies that $u|_{\bar{G}_1 \cup \bar{G}_1' \cup \bar{G}_1'' \cup \bar{G}_1'''} = 0$.

<div align="right">Q.E.D.</div>

3   Exterior Bitsadze-Lavrentjev problem: Frankl Case

Consider the general Bitsadze-Lavrentjev quaterelliptic - quaterhyperbolic equation (1.2) with eight parabolic lines of degeneracy in a bounded doubly connected mixed domain $\tilde{D}$ with a piecewise smooth boundary

$$\partial\tilde{D} = Ext\left(\tilde{D}\right) \cup Int\left(\tilde{D}\right) = \left[ExtEl\left(\tilde{D}\right) \cup ExtHn\left(\tilde{D}\right)\right] \cup Int\left(\tilde{D}\right),$$

where $\tilde{D}$ is a part of D, and $Int\left(\tilde{D}\right) = Int\left(D\right)$, as well as

$$ExtEl\left(\tilde{D}\right) = \Gamma_0 \cup \Gamma_0' \cup \Gamma_0'' \cup \Gamma_0''',$$

is the elliptic exterior boundary of $\tilde{D}$ and

$$ExtHn\left(\tilde{D}\right) = \left(\tilde{\Gamma}_2 \cup \tilde{\Gamma}_2'\right) \cup (\tilde{\gamma}_2 \cup \tilde{\gamma}_2') \cup \left(\tilde{\Delta}_1 \cup \tilde{\Delta}_1'\right) \cup \left(\tilde{\delta}_1 \cup \tilde{\delta}_1'\right),$$

is the non-characteristic hyperbolic exterior boundary of $\tilde{D}$, such that:

$$Ext\left(\tilde{D}\right) = ExtEl\left(\tilde{D}\right) \cup ExtHn\left(\tilde{D}\right) = \left(\Gamma_0 \cup \Gamma_0' \cup \Gamma_0'' \cup \Gamma_0'''\right)$$
$$\cup \left[\left(\tilde{\Gamma}_2 \cup \tilde{\Gamma}_2'\right) \cup (\tilde{\gamma}_2 \cup \tilde{\gamma}_2') \cup \left(\tilde{\Delta}_1 \cup \tilde{\Delta}_1'\right) \cup \left(\tilde{\delta}_1 \cup \tilde{\delta}_1'\right)\right],$$

is the exterior boundary of $\tilde{D}$, with the following non-characteristics $\tilde{\Gamma}_2, \tilde{\Gamma}_2', \tilde{\gamma}_2, \tilde{\gamma}_2', \tilde{\Delta}_1, \tilde{\Delta}_1', \tilde{\delta}_1, \tilde{\delta}_1'$:

$$\tilde{\Gamma}_2 : \sqrt{M\left(x\right)}dx \geq \sqrt{-K\left(y\right)}dy > 0; \quad \tilde{\Delta}_1' : \sqrt{M\left(x\right)}dx \leq \sqrt{-K\left(y\right)}dy < 0;$$
$$\tilde{\Gamma}_2' : \sqrt{M\left(x\right)}dx \leq -\sqrt{-K\left(y\right)}dy \leq 0; \quad \tilde{\Delta}_1 : \sqrt{M\left(x\right)}dx \geq -\sqrt{-K\left(-y\right)}dy > 0;$$
$$\tilde{\gamma}_2 : 0 \geq \sqrt{-M\left(x\right)}dx \geq \sqrt{K\left(y\right)}dy; \quad \tilde{\delta}_1' : 0 \leq \sqrt{-M\left(x\right)}dx \leq \sqrt{K\left(y\right)}dy;$$
$$\tilde{\gamma}_2' : 0 \geq \sqrt{-M\left(x\right)}dx \geq -\sqrt{K\left(y\right)}dy; \quad \tilde{\delta}_1 : 0 \leq \sqrt{-M\left(x\right)}dx \leq -\sqrt{K\left(y\right)}dy,$$

or

$$\tilde{\Gamma}_2 \cup \tilde{\Delta}_1' : 0 \leq \frac{dy}{dx}\left(= \frac{\nu_1}{-\nu_2}\right) \leq \frac{\sqrt{M\left(x\right)}}{\sqrt{-K\left(y\right)}}; \quad \tilde{\Gamma}_2' \cup \tilde{\Delta}_1 : 0 \geq \frac{dy}{dx} \geq -\frac{\sqrt{M\left(x\right)}}{\sqrt{-K\left(y\right)}};$$
$$\tilde{\gamma}_2 \cup \tilde{\delta}_1' : \frac{dy}{dx} \geq \frac{\sqrt{-M\left(x\right)}}{\sqrt{K\left(y\right)}}; \quad \tilde{\gamma}_2' \cup \tilde{\delta}_1 : \frac{dy}{dx} \leq -\frac{\sqrt{-M\left(x\right)}}{\sqrt{K\left(y\right)}},$$

satisfying the non-characteristic relation $K\left(y\right)\left(dy\right)^2 + M\left(x\right)\left(dx\right)^2 > 0$, and intersecting characteristics $\Gamma_1, \Gamma_1', \gamma_1, \gamma_1', \Delta_2, \Delta_2', \delta_2, \delta_2'$, only once. Let us consider the intersection points of the hyperbolic characteristics: $\tilde{\Gamma}_2 \cap \tilde{\Gamma}_2' = \left\{\tilde{P}_2\right\}$, where $\tilde{P}_2 = \left(\tilde{x}_2, 1/2\right), 1 < \tilde{x}_2 < 3/2$; $\tilde{\Delta}_1 \cap \tilde{\Delta}_1' = \left\{\tilde{P}_1'\right\}$, where $\tilde{P}_1' = \left(\tilde{x}_1', 1/2\right), -5/2 < \tilde{x}_1' < -2$; $\tilde{\gamma}_2 \cap \tilde{\gamma}_2' = \left\{\tilde{Q}_2\right\}$, where $\tilde{Q}_2 = \left(-1/2, \tilde{y}_2\right), 2 < \tilde{y}_2 < 5/2$;

$\tilde{\delta}_1 \cap \tilde{\delta}'_1 = \left\{ \tilde{Q}'_1 \right\}$ where $\tilde{Q}'_1 = (-1/2, \tilde{y}'_1), -3/2 < \tilde{y}'_1 < -1$. Let the right hyperbolic domain $\tilde{G}_2 \subset G_2 = \{(x, y) \in D : 0 < x < 2, 0 < y < 1\}$ with boundary

$$\partial \tilde{G}_2 = (O_1 B_1) \cup (O_2 B_2) \cup (\Gamma_1 \cup \Gamma'_1) \cup \left(\tilde{\Gamma}_2 \cup \tilde{\Gamma}'_2 \right).$$

Let the upper hyperbolic domain $\tilde{G}'_2 \subset G'_2 = \{(x, y) \in D : -1 < x < 0, 1 < y < 3\}$ with boundary

$$\partial \tilde{G}'_2 = (O_1 Z_1) \cup (O'_1 E_1) \cup (\gamma_1 \cup \gamma'_1) \cup (\tilde{\gamma}_2 \cup \tilde{\gamma}'_2).$$

Let the left hyperbolic domain $\tilde{G}''_2 \subset G''_2 = \{(x, y) \in D : -3 < x < -1, 0 < y < 1\}$ with boundary

$$\partial \tilde{G}''_2 = (O'_1 A_1) \cup (O'_2 A_2) \cup \left(\tilde{\Delta}_1 \cup \tilde{\Delta}'_1 \right) \cup (\Delta_2 \cup \Delta'_2).$$

Let the lower hyperbolic domain $\tilde{G}'''_2 \subset G'''_2 = \{(x, y) \in D : -1 < x < 0, -2 < y < 0\}$ with boundary

$$\partial \tilde{G}'''_2 = (O_2 Z_2) \cup (O'_2 E_2) \cup \left(\tilde{\delta}_1 \cup \tilde{\delta}'_1 \right) \cup (\delta_2 \cup \delta'_2).$$

Assume boundary conditions on the above exterior boundary $Ext\left(\tilde{D}\right)$:

$$u = \begin{cases} \varphi_1(s) & on & \Gamma_0 & ; & \varphi_2(s) & on & \Gamma'_0 \\ \varphi_3(s) & on & \Gamma''_0 & ; & \varphi_4(s) & on & \Gamma'''_0 \\ \tilde{\psi}_1(x) & on & \tilde{\Gamma}_2 & ; & \tilde{\psi}_2(x) & on & \tilde{\Gamma}'_2 \\ \tilde{\psi}_3(x) & on & \tilde{\gamma}_2 & ; & \tilde{\psi}_4(x) & on & \tilde{\gamma}'_2 \\ \tilde{\psi}_5(x) & on & \tilde{\Delta}_1 & ; & \tilde{\psi}_6(x) & on & \tilde{\Delta}'_1 \\ \tilde{\psi}_7(x) & on & \tilde{\delta}_1 & ; & \tilde{\psi}_8(x) & on & \tilde{\delta}'_1 \end{cases} \tag{3.1}$$

with continuous prescribed values.

The exterior Bitsadze-Lavrentjev problem: Frankl Case or Problem (BL-F):

consists of finding a solution u of the quaterelliptic -quaterhyperbolic equation (1.2) with eight parabolic lines in $\tilde{D} (\subset D)$ and which assumes continuous prescribed values (3.1).

Theorem 3.1. (UNIQUENESS THEOREM): Consider the quaterelliptic - quaterhyperbolic equation (1.2) with eight parabolic lines and the boundary condition (3.1). Assume the above mixed doubly connected domain $\tilde{D} (\subset D)$ and the following conditions:

$(R_1) \quad r \leq 0$ on the interior boundary $Int\left(\tilde{D}\right) (= Int(D))$,

$(R_2) \quad \begin{cases} xdy - (y-1)dx \geq 0 & on \quad \Gamma_0 & ; & xdy - ydx \geq 0 & on \quad \Gamma'_0 \\ (x+1)dy - (y-1)dx \geq 0 & on \quad \Gamma''_0 & ; & (x+1)dy - ydx \geq 0 & on \quad \Gamma'''_0 \end{cases}$,

$(R_3) \quad \begin{cases} 2r + xr_x + (y-1)r_y \leq 0 & in \quad G_1 & ; & r + xr_x \leq 0 & in \quad \tilde{G}_2 \\ 2r + xr_x + yr_y \leq 0 & in \quad G'_1 & ; & r + (y-1)r_y \leq 0 & in \quad \tilde{G}'_2 \\ 2r + (x+1)r_x + (y-1)r_y \leq 0 & in \quad G''_1 & ; & r + (x+1)r_x \leq 0 & in \quad \tilde{G}''_2 \\ 2r + (x+1)r_x + yr_y \leq 0 & in \quad G'''_1 & ; & r + yr_y \leq 0 & in \quad \tilde{G}'''_2 \end{cases}$,

$(R_4) \quad K = M = +1, \quad in \quad G_1 \cup G'_1 \cup G''_1 \cup G'''_1,$

(R_5) $\begin{cases} K = -1 \quad , \quad M = +1 \quad in \quad \tilde{G}_2 \cup \tilde{G}_2'' \\ K = +1 \quad , \quad M = -1 \quad in \quad \tilde{G}_2' \cup \tilde{G}_2''' \end{cases}$.

Let $()_x = \partial\,()/\partial x$, $()^\bullet = d\,()/dx$, $()_y = \partial\,()/\partial y$, $()' = d\,()/dy$, where $f = f(x,y)$ is continuous in $\tilde{D}\,(\subset D)$, $r = r(x,y)$ is once-continuously differentiable in $\tilde{D}\,(\subset D)$, $K = K(y)) := sgn(y(y - 1)|k(y)|)$ is a discontinuous function of y, for $y \in [-k_1, k_2]: -k_1 = inf\{y : (x,y) \in \tilde{D}\,(\subset D)\}$ and $k_2 = sup\{y : (x,y) \in \tilde{D}\,(\subset D)\}$, and $M = M(x) := sgn(x(x + 1)|m(x)|)$ is discontinuous for $x \in [-m_1, m_2]: -m_1 = inf\{x : (x,y) \in \tilde{D}\,(\subset D)\}$ and $m_2 = sup\{x : (x,y) \in \tilde{D}\,(\subset D)\}$, such that $k = k(y) \neq 0$; $m = m(x) \neq 0$, everywhere in D, as well as

$$K = \begin{cases} +1 & for & \{y < 0\} \cup \{y > 1\} \\ 0 & for & \{y = 0\} \cup \{y = 1\} \\ -1 & for & \{0 < y < 1\} \end{cases} \quad ; \quad M = \begin{cases} +1 & for & \{x < -1\} \cup \{x > 0\} \\ 0 & for & \{x = 0\} \cup \{x = -1\} \\ -1 & for & \{-1 < x < 0\} \end{cases} .$$

Then Problem (BL-F) has at most one quasi-regular solution in $\tilde{D}\,(\subset D)$.

Proof. We apply the well-known energy integral method , and employ the above Bitsadze-Lavrentjev equation (1.2) and boundary condition (3.1). First, we assume two quasi-regular solutions u_1, u_2 of Problem (BL-F). Then we claim that $u = u_1 - u_2 = 0$ holds in the domain $\tilde{D}\,(\subset D)$. In fact, we investigate

$$0 = \tilde{J} = 2\left\langle \tilde{l}u, Lu \right\rangle_0 = \iint\limits_D 2\tilde{l}uLudxdy,$$

where $\tilde{l}u = \tilde{b}\,(x)\,u_x + \tilde{c}\,(y)\,u_y$, and $Lu = L(u_1 - u_2) = Lu_1 - Lu_2 = f - f = 0$ in $\tilde{D}\,(\subset D)$, with choices

$$\tilde{b} = \tilde{b}\,(x) = \begin{cases} x & in & G_1 \cup G_1' \cup \tilde{G}_2 \\ x + 1 & in & G_1'' \cup G_1''' \cup \tilde{G}_2'' \\ 0 & in & \tilde{G}_2' \cup \tilde{G}_2''' \end{cases} ,$$

and

$$\tilde{c} = \tilde{c}\,(y) = \begin{cases} y & in & G_1' \cup G_1''' \cup \tilde{G}_2''' \\ y - 1 & in & G_1 \cup G_1'' \cup \tilde{G}_2' \\ 0 & in & \tilde{G}_2 \cup \tilde{G}_2'' \end{cases} .$$

The rest of the proof is similar to the proof of the uniqueness theorem 2.2. Q.E.D.

For the exterior Tricomi problem, except clearly proving in additional that the following condition holds on the non-characteristic hyperbolic exterior boundary

$$ExtHn\left(\tilde{D}\right) = \left(\tilde{\Gamma}_2 \cup \tilde{\Gamma}_2'\right) \cup (\tilde{\gamma}_2 \cup \tilde{\gamma}_2') \cup \left(\tilde{\Delta}_1 \cup \tilde{\Delta}_1'\right) \cup \left(\tilde{\delta}_1 \cup \tilde{\delta}_1'\right) :$$

$$0 < \tilde{b}\nu_1 + \tilde{c}\nu_2 = \begin{cases} x\nu_1 & on & \tilde{\Gamma}_2 \cup \tilde{\Gamma}_2' \\ (y - 1)\,\nu_2 & on & \tilde{\gamma}_2 \cup \tilde{\gamma}_2' \\ (x + 1)\,\nu_1 & on & \tilde{\Delta}_1 \cup \tilde{\Delta}_1' \\ y\nu_2 & on & \tilde{\delta}_1 \cup \tilde{\delta}_1' \end{cases} .$$

We note that $0 = J_{\tilde{D}} = I_{\tilde{D}} + I_{\partial \tilde{D}}$; $I_{\tilde{D}} \geq 0$; $I_{\partial \tilde{D}} \geq 0$. In fact,

$$
\begin{aligned}
I_{\tilde{D}} = &-\iint\limits_{G_1} \left(2r + x r_x + (y-1) r_y\right) u^2 dx dy - \iint\limits_{G_1'} \left(2r + x r_x + y r_y\right) u^2 dx dy \\
&- \iint\limits_{G_1''} \left(2r + (x+1) r_x + (y-1) r_y\right) u^2 dx dy \\
&- \iint\limits_{G_1'''} \left(2r + (x+1) r_x + y r_y\right) u^2 dx dy + \iint\limits_{\tilde{G}_2 \cup \tilde{G}_2' \cup \tilde{G}_2'' \cup \tilde{G}_2'''} \left(u_x^2 + u_y^2\right) dx dy \\
&- \iint\limits_{\tilde{G}_2} \left(r + x r_x\right) u^2 dx dy - \iint\limits_{\tilde{G}_2'} \left(r + (y-1) r_y\right) u^2 dx dy \\
&- \iint\limits_{\tilde{G}_2''} \left(r + (x+1) r_x\right) u^2 dx dy - \iint\limits_{\tilde{G}_2'''} \left(r + y r_y\right) u^2 dx dy \quad \geq \quad 0.
\end{aligned}
$$

Also

$$
\begin{aligned}
I_{\partial \tilde{D}} = &I_{\partial D} + \left[\left(I_{\tilde{\Gamma}_2} + I_{\tilde{\Gamma}_2'}\right) + \left(I_{\tilde{\gamma}_2} + I_{\tilde{\gamma}_2'}\right) + \left(I_{\tilde{\Delta}_1} + I_{\tilde{\Delta}_1'}\right) + \left(I_{\tilde{\delta}_1} + I_{\tilde{\delta}_1'}\right)\right] \\
= &\int\limits_{\Gamma_0} \left(x \nu_1 + (y-1) \nu_2\right) \left(\nu_1^2 + \nu_2^2\right) N^2 ds \\
&+ \int\limits_{\Gamma_0'} \left(x \nu_1 + y \nu_2\right) \left(\nu_1^2 + \nu_2^2\right) N^2 ds + \int\limits_{\Gamma_0''} \left((x+1) \nu_1 + (y-1) \nu_2\right) \left(\nu_1^2 + \nu_2^2\right) N^2 ds \\
&+ \int\limits_{\Gamma_0'''} \left((x+1) \nu_1 + y \nu_2\right) \left(\nu_1^2 + \nu_2^2\right) N^2 ds + \int\limits_{\Gamma_1} x \left[(u_x - u_y)^2 - r u^2\right] dx \\
&- \int\limits_{\Gamma_1'} x \left[(u_x + u_y)^2 - r u^2\right] dx + \int\limits_{\gamma_1} (y-1) \left[(u_x - u_y)^2 - r u^2\right] dx
\end{aligned}
$$

$$+ \int_{\gamma_1'} (y-1) \left[(u_x + u_y)^2 - ru^2 \right] dx - \int_{\Delta_2} (x+1) \left[(u_x + u_y)^2 - ru^2 \right] dx$$

$$+ \int_{\Delta_2'} (x+1) \left[(u_x - u_y)^2 - ru^2 \right] dx + \int_{\delta_2} y \left[(u_x + u_y)^2 - ru^2 \right] dx$$

$$+ \int_{\delta_2'} y \left[(u_x - u_y)^2 - ru^2 \right] dx \ + \int_{\tilde{\Gamma}_2 \cup \tilde{\Gamma}_2'} x\nu_1 \left(-\nu_1^2 + \nu_2^2 \right) N^2 ds$$

$$+ \int_{\tilde{\gamma}_2 \cup \tilde{\gamma}_2'} (y-1) \nu_2 \left(\nu_1^2 - \nu_2^2 \right) N^2 ds + \int_{\tilde{\Delta}_1 \cup \tilde{\Delta}_1'} (x+1) \nu_1 \left(-\nu_1^2 + \nu_2^2 \right) N^2 ds$$

$$+ \int_{\tilde{\delta}_1 \cup \tilde{\delta}_1'} y\nu_2 \left(\nu_1^2 - \nu_2^2 \right) N^2 ds \ \geq 0,$$

because of the following conditions:

$$x\nu_1|_{\tilde{\Gamma}_2 \cup \tilde{\Gamma}_2'} \geq 0 \ ; \ (y-1)\nu_2|_{\tilde{\gamma}_2 \cup \tilde{\gamma}_2'} \geq 0 \ ; \ (x+1)\nu_1|_{\tilde{\Delta}_1 \cup \tilde{\Delta}_1'} \geq 0 \ ; \ y\nu_2|_{\tilde{\delta}_1 \cup \tilde{\delta}_1'} \geq 0,$$

as well as:

$$\left(-\nu_1^2 + \nu_2^2 \right)|_{\left(\tilde{\Gamma}_2 \cup \tilde{\Gamma}_2' \right) \cup \left(\tilde{\Delta}_1 \cup \tilde{\Delta}_1' \right)} \geq 0 \ ; \ \left(\nu_1^2 - \nu_2^2 \right)|_{\left(\tilde{\gamma}_2 \cup \tilde{\gamma}_2' \right) \cup \left(\tilde{\delta}_1 \cup \tilde{\delta}_1' \right)} \geq 0.$$

We note that Γ_2 has an analogous equation as Γ_1', because Γ_2 is parallel to Γ_1'.

On the non-characteristic $\tilde{\Gamma}_2 : \ u = 0$, one obtains

$$I_{\tilde{\Gamma}_2} = - \int_{\tilde{\Gamma}_2} x \left[\left(u_x^2 + u_y^2 \right) dy + 2u_x u_y dx - ru^2 dy \right] \ \geq \ 0,$$

as due to $u|_{\tilde{\Gamma}_2} = 0 \ : \ u_x = \nu_1 N \ , \ u_y = \nu_2 N, \ N = normalizing \ factor$, such that

$$\left(u_x^2 + u_y^2 \right) dy + 2u_x u_y dx - ru^2 dy = \left[\left(\nu_1^2 + \nu_2^2 \right) \nu_1 + 2\nu_1 \nu_2 \left(-\nu_2 \right) - r.0 \right] N^2 ds$$

$$= - \nu_1 \left(-\nu_1^2 + \nu_2^2 \right) N^2 ds \ \geq \ 0,$$

$$-\nu_1^2 + \nu_2^2 = - \left(dy/ds \right)^2 + \left(-dx/ds \right)^2 = \left(dx - dy \right) \left(dx + dy \right) / \left(ds \right)^2 \ \geq \ 0,$$

where $0 < dy \leq dx$ on $\tilde{\Gamma}_2$, and $\nu_1|_{\tilde{\Gamma}_2} < 0$.

Similarly on the following non-characteristics: $\tilde{\Gamma}_2', \tilde{\Delta}_1, \tilde{\Delta}_1' : u = 0$,
where $x\nu_1|_{\tilde{\Gamma}_2 \cup \tilde{\Gamma}_2'} \geq 0 \ ; \ (x+1)\nu_1|_{\tilde{\Delta}_1 \cup \tilde{\Delta}_1'} \geq 0 \ ,$
as well as: $\left(-\nu_1^2 + \nu_2^2 \right)|_{\left(\tilde{\Gamma}_2 \cup \tilde{\Gamma}_2' \right) \cup \left(\tilde{\Delta}_1 \cup \tilde{\Delta}_1' \right)} \geq 0 \ .$

On the other hand, on the non-characteristic $\tilde{\gamma}_2 : u = 0$, we find

$$I_{\tilde{\gamma}_2} = \int_{\tilde{\gamma}_2} (y-1) \left[\left(u_x^2 + u_y^2 \right) dx + 2u_x u_y dy - ru^2 dx \right] \ \geq \ 0,$$

as due to $u|_{\tilde{\gamma}_2} = 0$: $u_x = \nu_1 N$, $u_y = \nu_2 N$, $N = normalizing\,factor$, such that

$$\left(u_x^2 + u_y^2\right) dx + 2u_x u_y dy - ru^2 dx = \left[\left(\nu_1^2 + \nu_2^2\right)(-\nu_2) + 2\nu_1\nu_2(\nu_1) - r.0\right] N^2 ds$$
$$= \nu_2\left(\nu_1^2 - \nu_2^2\right) N^2 ds \geq 0,$$

$$\nu_1^2 - \nu_2^2 = (dy/ds)^2 - (-dx/ds)^2 = (dy - dx)(dy + dx)/(ds)^2 \geq 0,$$

where $dy \leq dx < 0$ on $\tilde{\gamma}_2$, and $\nu_2|_{\tilde{\gamma}_2} > 0$.

Similarly on the following non-characteristics: $\tilde{\gamma}_2'$, $\tilde{\delta}_1$, $\tilde{\delta}_1'$: $u = 0$,

where $(y-1)\nu_2|_{\tilde{\gamma}_2 \cup \tilde{\gamma}_2'} \geq 0$; $y\nu_2|_{\tilde{\delta}_1 \cup \tilde{\delta}_1'} \geq 0$,

as well as: $\left(\nu_1^2 - \nu_2^2\right)|_{\left(\tilde{\gamma}_2 \cup \tilde{\gamma}_2'\right) \cup \left(\tilde{\delta}_1 \cup \tilde{\delta}_1'\right)} \geq 0.$

4 Open problems

4.1. Extend "quasi-regularity" of solutions to "regularity" by fixing singularities at the following twelve "singular" points:

$$O_1 = (0,1)\,,\; O_1' = (-1,1)\,,\; O_2 = (0,0)\,,\; O_2' = (-1,0)\,;$$

$$A_1 = (-3,1)\,,\; B_1 = (2,1)\,,\; A_2 = (-3,0)\,,\; B_2 = (2,0)\,;$$

$$E_1 = (-1,3)\,,\; Z_1 = (0,3)\,,\; E_2 = (-1,-2)\,,\; Z_2 = (0,-2)\,.$$

Then find the formula for these regular solutions.

4.2. Investigate the exterior Bitsadze-Lavrentjev problem in a multiply connected mixed domain.

4.3. Establish "well-posedness" of solutions for the 3D exterior Bitsadze-Lavrentjev problem , in the sense that there is at most one quasi-regular solution and a weak solution exists.

4.4. Solve the n - dimensional Bitsadze-Lavrentjev problem in a multiply connected mixed domain.

4.5. Establish the extremum principle for the exterior Bitsadze-Lavrentjev problem : "A solution of the exterior Bitsadze-Lavrentjev problem , vanishing on the exterior boundary of the considered mixed domain, achieves neither a positive maximum nor a negative minimum on open arcs of the type-degeneracy curves."

4.6. Solve the Bitsadze-Lavrentjev problem for Partial Differential Equations of second order:

4.6.1. $\operatorname{sgn}\left(x^2 + y^2 - 1\right) u_{xx} + u_{yy} + r(x,y)u = f(x,y);$

4.6.2. $\operatorname{sgn}\left(y^3 - y\right) u_{xx} + \operatorname{sgn}\left(x^3 - x\right) u_{yy} + r(x,y)u = f(x,y);$

4.6.3. $\operatorname{sgn}\left(x^{2/3} + y^{2/3} - 1\right) u_{xx} + u_{yy} + r(x,y)u = f(x,y)\,;$

4.6.4. $\operatorname{sgn}\left((y - x^m)(y - x^n)\right) u_{xx} + u_{yy} + r(x,y)u = f(x,y)\,;$

4.6.5. $\operatorname{sgn}\left(y - x^n\right) u_{xx} + \operatorname{sgn}\left(x - y^m\right) u_{yy} + r(x,y)u = f(x,y)\,;$

4.6.6. $\operatorname{sgn}\left(y^k - x^m \pm x^n\right) u_{xx} + \operatorname{sgn}\left(x^k - y^m \pm y^n\right) u_{yy} + r(x,y)u = f(x,y)\,;$

4.6.7. $\operatorname{sgn}\left(y^m(y - x^n)\right) u_{xx} + \operatorname{sgn}\left(x^m(x - y^n)\right) u_{yy} + r(x,y)u = f(x,y)\,;$

4.6.8. $\operatorname{sgn}\left((y - x^m)(y - x^n)\right) u_{xx} + \operatorname{sgn}\left((x - y^\alpha)(x - y^\beta)\right) u_{yy} + ru = f(x,y)\,.$

4.7. Solve the Bitsadze-Lavrentjev problem for PDE of fourth order:

$$\left(\operatorname{sgn} \left(y - x^l \right) \frac{\partial^2}{\partial x^2} + \operatorname{sgn} \left(x - y^n \right) \frac{\partial^2}{\partial y^2} + r \right)^2 u = f.$$

4.8. Solve the 3 - dimensional Bitsadze-Lavrentjev problem for mixed type PDE of second order:

$$\operatorname{sgn} \left(z \right) \left(u_{xx} \pm u_{yy} \right) + \operatorname{sgn} \left(xy \right) u_{zz} + ru = f.$$

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